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KILLING TENSORS AND A NEW GEOMETRIC DUALITY

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Abstract

We present a theorem describing a dual relation between the local geometry of a space admitting a symmetric second-rank Killing tensor, and the local geometry of a space with a metric specified by this Killing tensor. The relation can be generalized to spinning spaces, but only at the expense of introducing torsion. This introduces new supersymmetries in their geometry. Interesting examples in four dimensions include the Kerr-Newman metric of spinning black-holes and self-dual Taub-NUT.

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1 Introduction

The importance of symmetries in the description of physical systems can hardly be over-estimated. In the case of dynamical systems in particular, continuous symmetries determine the structure of the algebra of observables by Noether's theorem, giving rise to constants of motion in classical mechanics and quantum numbers labeling stationary states in quantum theory.

In a geometrical setting symmetries are connected with isometries associated with Killing vectors and, more generally, Killing tensors on the configuration space of the system. An example is the motion of a point particle in a space with isometries [1], which is a physicist's way of studying the geodesic structure of a manifold. Contact with the algebraic approach is made through Lie-derivatives and their commutators. In [1, 2] such studies were extended to spinning space-times described by supersymmetric extensions of the geodesic motion, and in [3] it was shown that this can give rise to interesting new types of supersymmetry as well.

This paper concerns spaces on which there exists a symmetric second-rank Killing tensor, including such interesting cases as the four-dimensional Kerr-Newman space-time describing spinning and/or charged black holes, and four-dimensional Taub-NUT which describes a gravitational instanton [4] and appears as a low-energy effective action for monopole scattering [5, 6]. These second-rank Killing tensors correspond to constants of motion which are quadratic in the momenta and play an important role in the complete solution of the problem of geodesic motion in these spaces [7]. Similar results concerning the monopole with spin and 3-D fermion systems have been presented in [8, 9].

The main aim of the paper is to present and illustrate a theorem concerning the reciprocal relation between two local geometries described by metrics which are Killing tensors with respect to one another. In sect. 2 the basic theorem is presented. It is illustrated in sect. 3 with the four-dimensional examples mentioned above. In sect. 4 we extend the discussion to the motion of particles with spin and charge in curved space, including torsion and electro-magnetic background fields. This generalizes results obtained by Tanimoto [10]. In sect. 5 the relation between Killing tensors and new supersymmetries, representing a certain square root of the associated constants of motion, is explained. The basic ingredient is the existence of so-called Killing-Yano tensors [3]. The duality theorem is then generalized to include geometries with Killing-Yano tensors, and it is shown that in general this requires the introduction of torsion. Finally the explicit expressions for the Killing-Yano tensors and the associated torsion-tensors for the examples of Kerr-Newman and Taub-NUT are given.

2 Dual geometries

Suppose a space (of either Euclidean or Lorentzian signature) with metric $g_{\mu\nu}(x)$ admits a second-rank Killing tensor field $K_{\mu\nu}(x)$:

$$K_{(\mu\nu;\lambda)} = 0. \tag{1}$$

Here the semi-colon denotes a Riemannian covariant derivative, and the parentheses denote complete symmetrization over the component indices. The equation of motion of a particle on a geodesic is derived from the action

$$S = \int d\tau \left(\frac{1}{2}\, g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right). \tag{2}$$

The corresponding world-line hamiltonian is constructed in terms of the inverse (contravariant) metric $g^{\mu\nu}$:

$$H = \frac{1}{2}\, g^{\mu\nu} p_\mu p_\nu, \tag{3}$$

and the elementary Poisson brackets are

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu. \tag{4}$$

The equation of motion for a phase space function $F(x,p)$ can be computed from the Poisson brackets with the hamiltonian:

$$\frac{dF}{d\tau} = \{F, H\}. \tag{5}$$

From the contra-variant components $K^{\mu\nu}$ of the Killing tensor one can construct a constant of motion K:

$$K = \frac{1}{2}\, K^{\mu\nu} p_\mu p_\nu. \tag{6}$$

Its bracket with the hamiltonian vanishes precisely because of the condition (1). The constant of motion K generates symmetry transformations on the phase space linear in momentum:

$$\{x^\mu, K\} = K^{\mu\nu} p_\nu. \tag{7}$$

The infinitesimal transformation with parameter α can also be written in terms of the velocity as

$$\delta(\alpha)\, x^\mu = \alpha\, K^\mu_{\ \nu} \dot{x}^\nu. \tag{8}$$

The formal similarity between the constants of motion H and K, and the symmetrical nature of the condition implying the existence of the Killing tensor:

$$\{H, K\} = 0, \tag{9}$$

amount to a reciprocal relation between two different models: the model with hamiltonian H and constant of motion K, and a model with constant of motion H and hamiltonian K. In the second model H generates a symmetry in the phase space of the system. Thus the relation between the two models has a geometrical interpretation: it implies that if $K^{\mu\nu}$ are the contravariant components of a Killing tensor with respect to the inverse metric $g^{\mu\nu}$, then $g^{\mu\nu}$ must represent a Killing tensor with respect to the inverse metric defined by $K^{\mu\nu}$.

A more direct proof of this result can be given purely in terms of geometrical quantities. Given the metric $g_{\mu\nu}$ and the Killing tensor $K_{\mu\nu}$ satisfying eq.(1), we identify the contravariant components of K with a new contravariant metric $\tilde{g}^{\mu\nu}$:

$$\tilde{g}^{\mu\nu} \equiv K^{\mu\nu} = g^{\mu\lambda} K_{\lambda\kappa} g^{\kappa\nu}. \tag{10}$$

If these components form a non-singular $(d \times d)$-dimensional matrix, we denote its inverse by $\tilde{g}_{\mu\nu}$:

$$\tilde{g}^{\mu\lambda} \tilde{g}_{\lambda\nu} = \delta^{\mu}_{\nu}. \tag{11}$$

Now we can interpret $\tilde{g}_{\mu\nu}$ as the metric of another space. Define the associated Riemann-Christoffel connection $\tilde{\Gamma}_{\mu\nu}{}^{\lambda}$ as usual through the metric postulate

$$\tilde{D}_{\lambda} \tilde{g}_{\mu\nu} = 0. \tag{12}$$

Then it follows from eq.(1) that

$$\tilde{D}_{(\lambda} \tilde{K}_{\mu\nu)} = 0, \tag{13}$$

where $\tilde{K}_{\mu\nu}$ are the covariant components of g with respect to the metric $\tilde{g}$:

$$\tilde{K}_{\mu\nu} = \tilde{g}_{\mu\lambda} g^{\lambda\kappa} \tilde{g}_{\kappa\nu}. \tag{14}$$

This reciprocal relation between the metric structure and certain symmetries of pairs of spaces therefore constitutes a duality relation: performing the operation of mapping a Killing tensor to a metric twice leads back to the original geometry.

3 Examples: Kerr-Newman and Taub-NUT

Examples of manifolds with Killing-tensor fields include physically important ones, like the four-dimensional Kerr-Newman and Taub-NUT solutions of the Einstein or Einstein-Maxwell equations. In this section we present explicit expressions for their metric, Killing tensor and the metric of their dual spaces in the sense defined above.

Kerr-Newman. The Kerr-Newman geometry describes a charged spinning black hole; in a standard choice of co-ordinates the metric is given by the following line-element:

$$
\begin{aligned}
ds^2 &= -\frac{\Delta}{\rho^2}\left[dt - a\sin^2\theta\, d\varphi\right]^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\varphi - adt\right]^2 + \\
&\quad + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2,
\end{aligned}
\tag{15}
$$

Here

$$
\begin{aligned}
\Delta &= r^2 + a^2 - 2Mr + Q^2, \\
\rho^2 &= r^2 + a^2\cos^2\theta,
\end{aligned}
\tag{16}
$$

with Q the background electric charge, and $J = Ma$ the total angular momentum. The expression for ds^2 only describes the fields *outside* the horizon, which is located at

$$
r = M + \sqrt{M^2 - Q^2 - a^2}. \tag{17}
$$

The background electric charge also creates an electro-magnetic field described by the Maxwell 2-form

$$
\begin{aligned}
F &= \frac{Q}{\rho^4}(r^2 - a^2\cos^2\theta)dr \wedge \left[dt - a\sin^2\theta\, d\varphi\right] + \\
&+ \frac{2Qar\cos\theta\sin\theta}{\rho^4}d\theta \wedge \left[-adt + (r^2+a^2)d\varphi\right].
\end{aligned}
\tag{18}
$$

The Kerr-Newman metric admits a second-rank Killing tensor field, which can be described in this co-ordinate system by the quadratic form

$$
\begin{aligned}
K &= K_{\mu\nu}\, dx^\mu dx^\nu \\
&= \frac{a^2\cos^2\theta\Delta}{\rho^2}\left[dt - a\sin^2\theta\, d\varphi\right]^2 + \frac{r^2\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\varphi - adt\right]^2 \\
&\quad - \frac{\rho^2 a^2\cos^2\theta}{\Delta}dr^2 + r^2\rho^2 d\theta^2
\end{aligned}
\tag{19}
$$

Its contravariant components define the inverse metric $\tilde{g}^{\mu\nu}$ of the dual geometry. The dual line-element then becomes

$$
\begin{aligned}
d\tilde{s}^2 &= \tilde{g}_{\mu\nu} dx^\mu dx^\nu \\
&= \frac{\Delta}{\rho^2 a^2 \cos^2\theta} \left[dt - a \sin^2\theta \, d\varphi \right]^2 + \frac{\sin^2\theta}{\rho^2 r^2} \left[(r^2 + a^2) d\varphi - a dt \right]^2 \\
&\quad - \frac{\rho^2}{\Delta a^2 \cos^2\theta} dr^2 + \frac{\rho^2}{r^2} d\theta^2,
\end{aligned}
\tag{20}
$$

providing an explicit expression for the dual metric.

Taub-NUT. The four-dimensional Taub-NUT metric depends on a parameter m which can be positive or negative, depending on the application; for $m > 0$ it represents a non-singular solution of the self-dual Euclidean Einstein equation and as such is interpreted as a gravitational instanton. A standard form of the line-element is

$$
\begin{aligned}
ds^2 &= \left(1 + \frac{2m}{r}\right) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) \\
&\quad + \frac{4m^2}{1 + \frac{2m}{r}} \left(d\psi + \cos\theta d\varphi\right)^2 .
\end{aligned}
\tag{21}
$$

A symmetric Killing tensor with respect to this metric is represented by the quadratic form

$$
\begin{aligned}
K &= \left(1 + \frac{2m}{r}\right) \left(dr^2 + \frac{r^2}{m^2} (r+m)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) \\
&\quad + \frac{4m^2}{1 + \frac{2m}{r}} \left(d\psi + \cos\theta d\varphi\right)^2 .
\end{aligned}
\tag{22}
$$

The matrix inverse of the contravariant form of K gives the dual line element

$$
\begin{aligned}
d\tilde{s}^2 &= \left(1 + \frac{2m}{r}\right) \left(dr^2 + \frac{m^2 r^2}{(r+m)^2} \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) \\
&\quad + \frac{4m^2}{1 + \frac{2m}{r}} \left(d\psi + \cos\theta d\varphi\right)^2 .
\end{aligned}
\tag{23}
$$

The Taub-NUT metric admits three more second rank Killing tensors. They form a conserved vector of the Runge-Lenz type and are given by

$$
\begin{aligned}
K_{(i)} &= -\frac{2}{m}\left(1+\frac{2m}{r}\right)\left(1+\frac{m}{r}\right) r_i \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) \\
&\quad + \frac{8m^2}{r\left(1+\frac{2m}{r}\right)} r_i \left(d\psi + \cos\theta d\varphi\right)^2 + \frac{2r}{m}\left(1+\frac{2m}{r}\right)^2 dr dr_i \\
&\quad + 4\left(1+\frac{2m}{r}\right)\left(\vec{r}\times d\vec{r}\right)_i \left(d\psi + \cos\theta\varphi\right).
\end{aligned} \tag{24}
$$

These conserved quantities define the dual line-elements

$$
\begin{aligned}
d\tilde{s}^2_{(i)} &= \frac{-1}{r_i^2 - (r+2m)^2}\left\{-\frac{2m^2}{r}\left(1+\frac{2m}{r}\right) r_i \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)\right. \\
&\quad + \frac{8m^3\left(1+\frac{m}{r}\right)}{\left(1+\frac{2m}{r}\right)} r_i \left(d\psi + \cos\theta d\varphi\right)^2 + 2mr\left(1+\frac{2m}{r}\right)^2 dr dr_i \\
&\quad \left. + 4m^2\left(1+\frac{2m}{r}\right)\left(\vec{r}\times d\vec{r}\right)_i \left(d\psi + \cos\theta\varphi\right)\right\}.
\end{aligned} \tag{25}
$$

Note that in all examples the dual metrics are very similar, though not identical, to the original ones. The physical interpretation of the dual metrics remains to be clarified.

4 Electro-magnetism and torsion

The results obtained so far can be generalized to include spin and charge for the test particle. Electro-magnetic interactions are introduced via minimal coupling, whilst spin can be described by the supersymmetric extension of the dynamics, using a vector of Grassmann-odd co-ordinates ψ^μ. This also allows for torsion to be present, as is actually required for some of our purposes. As found in [3], the existence of Killing-tensors now requires new supersymmetric structures. The inclusion of charge was studied in [10].

Using these procedures, consider a charged, spinning particle which moves under the influence of an electro-magnetic field A_μ, a gravitational field $g_{\mu\nu}$, and a completely anti-symmetric torsion field A_{abc}, as described by the action

$$
\begin{aligned}
S &= \int d\tau \left\{\frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \frac{i}{2}\eta_{ab}\dot{\psi}^a\psi^b - \frac{i}{2}\dot{x}^\mu\psi^a\psi^b\left(\omega_{\mu ab} + A_{\mu ab}\right) + \right. \\
&\quad \left. + \dot{x}^\mu A_\mu - \frac{i}{2}\psi^a\psi^b F_{ab} - \frac{1}{4!}\psi^a\psi^b\psi^c\psi^d F_{abcd}\right\}.
\end{aligned} \tag{26}
$$

Here $\mu, \nu, \ldots$ label space-time coordinates, while $a, b, \ldots$ label local Lorentz coordinates. Both types of indices run from $1, \ldots, d$, the dimension of space-time. They can be converted into each other by contracting with an (inverse) vielbein ${e_a}^\mu(x)$ $({e_\mu}^a(x))$, defined by

$$ {e_a}^\mu {e_b}^\nu \eta^{ab} = g^{\mu\nu}. \tag{27} $$

The torsion will usually be written explicitly, except in a few cases where concise notation requires otherwise; this is always manifest in the text and the equations. In any case, $\omega_{\mu ab}$ is the bare spin connection, defined by

$$ \omega_{\mu ab} = {e_a}^\kappa e_{b[\mu,\kappa]} - {e_b}^\kappa e_{a[\mu,\kappa]} - {e_a}^\kappa {e_b}^\lambda e_{c\mu} e^c_{\ [\kappa,\lambda]}. \tag{28} $$

We will use square brackets to denote anti-symmetrisation and parentheses to denote symmetrisation over all indices enclosed, with total weight equal to one. Finally F_{ab} and F_{abcd} are contractions with vielbeins of the field strengths associated with A_μ and $A_{\mu\nu\kappa}$

$$ \begin{aligned} F_{\mu\nu} &= 2\partial_{[\mu} A_{\nu]}, \\ F_{\mu\nu\kappa\lambda} &= 4\partial_{[\mu} A_{\nu\kappa\lambda]} \end{aligned} \tag{29} $$

The classical equations of motion derived from the action (26) are

$$ g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} = F_{\mu\nu}\dot{x}^\nu - \frac{i}{2}\psi^a\psi^b R^T_{ab\mu\nu}\dot{x}^\nu - \frac{i}{2}\psi^a\psi^b D^T_\mu F_{ab} - \frac{1}{4}\psi^a\psi^b\psi^c\psi^d D^T_\mu F_{abcd}, \tag{30} $$

for the orbit of the particle, and

$$ \frac{D^T}{D\tau}\psi^a = F^a_{\ b}\psi^b - \frac{i}{3!} F^a_{\ bcd}\psi^a\psi^b\psi^c, \tag{31} $$

for the precession of the spin. In these equations we have by exception included torsion in the derivatives, as denoted by the superscript T, because this simplifies the expressions considerably.

The action (26) is invariant under the supersymmetry transformation

$$ \begin{aligned} \delta_\epsilon x^\mu &= -i\epsilon\psi^a {e_a}^\mu, \\ \delta_\epsilon \psi^a &= \epsilon \dot{x}^\mu {e_\mu}^a + i\epsilon\psi^b\psi^c {e_b}^\mu {\omega_{\mu c}}^a. \end{aligned} \tag{32} $$

This implies that there is a conserved quantity Q, the supercharge, which generates the transformation by its Dirac-bracket. The fundamental Dirac-brackts are

$$ \{x^\mu, P_\nu\} = \delta^\mu_\nu, \qquad \left\{\psi^a, \psi^b\right\} = -i\eta^{ab}, \tag{33} $$

where P_ν is the momentum conjugate to x^μ

$$P_\mu = \dot{x}^\nu g_{\mu\nu} - \frac{i}{2}\psi^a\psi^b\left(\omega_{\mu ab} + A_{\mu ab}\right) + A_\mu. \tag{34}$$

If, for notational convenience, we also define a (modified) covariant momentum Π_μ ($\tilde{\Pi}_\mu$) by

$$\begin{aligned} \Pi_\mu &= P_\mu + \frac{i}{2}\psi^a\psi^b\left(\omega_{\mu ab} + A_{\mu ab}\right) - A_\mu, \\ \tilde{\Pi}_\mu &= P_\mu + \frac{i}{2}\psi^a\psi^b\left(\omega_{\mu ab} + \frac{1}{3}A_{\mu ab}\right) - A_\mu, \end{aligned} \tag{35}$$

the supercharge Q and the Hamiltonian H are given by

$$\begin{aligned} Q &= \psi^a e_a{}^\mu \tilde{\Pi}_\mu, \\ H &= \frac{1}{2}\Pi_\mu\Pi_\nu g^{\mu\nu} + \frac{i}{2}\psi^a\psi^b F_{ab} + \frac{1}{4!}\psi^a\psi^b\psi^c\psi^d F_{abcd}. \end{aligned} \tag{36}$$

Using the brackets given by eq.(33) one can check that Q is indeed conserved

$$\{H, Q\} = 0, \tag{37}$$

and that

$$\{Q, Q\} = -2iH. \tag{38}$$

Eq.(38) implies eq.(37) by the Jacobi-identities.

The Poisson-Dirac brackets for general phase-space functions $F(x, \Pi, \psi)$ can be cast in a manifest covariant form:

$$\begin{aligned} \{F, G\} &= \mathcal{D}^T_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu}\mathcal{D}^T_\mu G \\ &+ \left(-\frac{i}{2}\psi^a\psi^b R^T_{ab\mu\nu} - 2A_{\mu\nu}{}^\lambda\Pi_\lambda + F_{\mu\nu}\right)\frac{\partial F}{\partial \Pi_\mu}\frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi^b}. \end{aligned} \tag{39}$$

with the covariant derivatives defined as

$$\mathcal{D}^T_\mu F = \partial_\mu F + \left(\omega_{\mu ab} + A_{\mu ab}\right)\psi^b \frac{\partial F}{\partial \psi_a} + \left(\Gamma_{\mu\nu}{}^\lambda + A_{\mu\nu}{}^\lambda\right)\Pi_\lambda \frac{\partial F}{\partial \Pi_\nu}. \tag{40}$$

and with a_F the Grassmann parity of F: $a_F = (0, 1)$ for $F = $ (even,odd). As before the superscript T denotes the inclusion of torsion. Note that the brackets satisfy the Ricci identity in the presence of torsion and electro-magnetism:

$$\{\Pi_\mu, \Pi_\nu\} \;=\; -\frac{i}{2}\psi^a\psi^b R^T_{ab\mu\nu} - 2A_{\mu\nu}{}^\lambda\Pi_\lambda + F_{\mu\nu}. \tag{41}$$

The covariant expression (39) for the brackets simplifies many calculations considerably.

5 New supersymmetries

The spinning particle model was constructed to be supersymmetric. Therefore, independent of the form of the metric there is always a conserved supercharge given by eq.(36). Of course it is possible that the model has more symmetry, but this will in general depend on the metric. In [3] it was found that the model admits an extra, generalised type of supersymmetry when the metric admits a tensor $f_{\mu\nu}$ satisfying

$$f_{\mu\nu} = -f_{\nu\mu}, \tag{42}$$

$$f_{\mu\kappa;\nu} + f_{\nu\kappa;\mu} = 0. \tag{43}$$

Such a tensor is called a Killing-Yano tensor. We will explain how this result was obtained and how it generalises to the case where electro-magnetism and torsion are present. Consider a quantity Q_f of the form

$$Q_f = \psi^a f_a{}^\mu \tilde{\Pi}_\mu + \frac{i}{3!}\psi^a\psi^b\psi^c c_{abc}. \tag{44}$$

This quantity is invariant under supersymmetry if

$$\{Q, Q_f\} = 0. \tag{45}$$

The Jacobi identities then guarantee that it is also conserved

$$\{H, Q_f\} = 0, \tag{46}$$

and hence it generates a symmetry of the action. Condition (45) imposes constraints on $f_a{}^\mu$ and c_{abc}. They read

$$e_a{}^{(\mu} f_b{}^{\nu)}\eta^{ab} = 0, \tag{47}$$

$$D^T_\mu f_\nu{}^a + D^T_\nu f_\mu{}^a = 0, \tag{48}$$

$$e_{[a}{}^\mu f_{b]}{}^\nu F_{\mu\nu} = 0, \tag{49}$$

$$c_{abc} = 2e_{[a}{}^\mu e_{b|}{}^\nu D_\nu f_{|c]\mu}. \tag{50}$$

Here D_μ denotes a derivative without torsion, while D^T_μ is a derivative with torsion. Eq.(47) says that $f^{\mu\nu} := e^{a\mu} f_a{}^\nu$ is anti-symmetric. Eq.(48) generalises the Killing-Yano equation (43) to the case with torsion. When there is an electro-magnetic field, there is an extra condition, eq.(49), on $f_a{}^\mu$ to define a conserved quantity Q_f. In [3], where the Killing-Yano tensor of the Kerr-Newman metric was considered, this last relation was not taken into account. However, it has been checked now that it is satisfied in that case. Once a tensor has been found which satisfies these three conditions, the tensor c_{abc} is defined by eq.(50) and a new Grassmann-odd constant of motion is obtained. This charge is not necessarily a second supercharge since in general

$$\{Q_f, Q_f\} = -2iZ \neq -2iH. \tag{51}$$

Hence we refer to it as a generalised supercharge. It is straightforward to check, using the Jacobi identities, that the only non-vanishing brackets between Q, Q_f, H and Z are (38) and (51).

6 Dual spinning spaces

In sect.2 it was argued, that the most direct route to the duality between metrics and Killing tensors comes from the symmetric Poisson-bracket relation

$$\{H, K\} = 0.$$

The Poisson-Dirac bracket (45) between ordinary and generalized supercharges:

$$\{Q, Q_f\} = 0$$

suggests a similar duality between the vielbein $e_a{}^\mu$ and the Killing-Yano tensor $f_a{}^\mu$, interchanging ordinary and generalized supersymmetry. In this section we show that this is indeed the case, provided that we allow explicitly for the presence of torsion.

Consider two spinning particle models, labeled I and II respectively, both describing a spinning particle interacting with gravitational, electro-magnetic and torsion fields. Let theory I also admit a Killing-Yano tensor $f_a{}^\mu$ satisfying eqs.(47) - (49), and let

$$f_a{}^\mu f_b{}^\nu \eta^{ab} = K^{\mu\nu}. \tag{52}$$

The symmetric tensor K on the right-hand side is a second-rank Killing tensor [3]. Hence if $K_{\mu\nu} \neq g_{\mu\nu}$, the metric of this space has a dual in the sense of sect.2. Note that in the present class of models there also exist examples in which $K_{\mu\nu} = g_{\mu\nu}$, even though $Q_f \neq Q$. This case corresponds to N-extended supersymmetry

$(N \geq 2)$. We might say that N-extended supersymmetric theories are self-dual in our geometric sense.

The spinning-particle theory is completely determined by its supercharge Q, as given by eq.(36). The term in Q proportional to P_μ defines the vielbein and thus the metric. This determines the spin connection, which implies that the torsion can be calculated from the terms proportional to $\psi^a \psi^b \psi^c$. Finally the electro-magnetic field is found from the term proportional to ψ^a.

Let us first introduce some notation. Theory I is defined by the vielbein ${e_a}^\mu$, the anti-symmetric torsion field A_{abc} and the electro-magnetic field A_μ. The momentum P_μ is given by eq.(34). For general functions F and G of the phase-space variables (x, P, ψ) the non-covariant form of the Poisson-Dirac bracket reads

$$\{F, G\}_I = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial P_\mu} - \frac{\partial F}{\partial P_\mu} \frac{\partial G}{\partial x^\mu} + i(-1)^{a_F} \frac{\partial F}{\partial \psi^a} \frac{\partial G}{\partial \psi_a}. \tag{53}$$

The theory has a supercharge Q and a generalised supercharge Q_f defined by

$$Q = \psi^a {e_a}^\mu P_\mu + \frac{i}{2} \psi^a \psi^b \psi^c \left({e_a}^\mu \omega_{\mu bc} + \frac{1}{3} A_{abc} \right) - \psi^a {e_a}^\mu A_\mu, \tag{54}$$

$$Q_f = \psi^a {f_a}^\mu P_\mu + \frac{i}{2} \psi^a \psi^b \psi^c \left({f_a}^\mu \omega_{\mu bc} + \frac{1}{3} {f_a}^\mu {e_\mu}^d A_{dbc} + \frac{1}{3} c_{abc} \right) - \psi^a {f_a}^\mu A_\mu, \tag{55}$$

with

$$\{Q, Q_f\}_I = 0. \tag{56}$$

The tensor c_{abc} is given by eq.(50).

Another spinning particle model, theory II, is defined by a vielbein ${\tilde{e}_a}{}^\mu$, an anti-symmetric torsion field $\tilde{A}_{abc}$ and an electro-magnetic field $\tilde{A}_\mu$. The momentum $\bar{P}_\mu$ is given by

$$\bar{P}_\mu = \dot{x}^\nu \tilde{g}_{\mu\nu} - \frac{i}{2} \psi^a \psi^b \left(\tilde{\omega}_{\mu ab} + \tilde{A}_{\mu ab} \right) + \tilde{A}_\mu. \tag{57}$$

Here $\tilde{g}_{\mu\nu}$ and $\tilde{\omega}_{\mu ab}$ are the metric and spin connection calculated from ${\tilde{e}_a}{}^\mu$ in a way analogous to eqs.(27) and (28). Since $\bar{P}_\mu \neq P_\mu$, theory II has different Dirac brackets

$$\{F, G\}_{II} = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial \bar{P}_\mu} - \frac{\partial F}{\partial \bar{P}_\mu} \frac{\partial G}{\partial x^\mu} + i(-1)^{a_F} \frac{\partial F}{\partial \psi^a} \frac{\partial G}{\partial \psi_a}. \tag{58}$$

However, if we define an operation $F \to \bar{F}$ by

$$F(x, P, \psi) \to \overline{F(x, P, \psi)} = F(x, \bar{P}, \psi), \tag{59}$$

it is straightforward to deduce

$$\left\{ \bar{F}, \bar{G} \right\}_{II} = \overline{\{F, G\}_I}. \tag{60}$$

Theory II has a supercharge as well. It is given by

$$\tilde{Q} = \psi^a \tilde{e}_a{}^\mu \bar{P}_\mu + \frac{i}{2} \psi^a \psi^b \psi^c \left(\tilde{e}_a{}^\mu \tilde{\omega}_{\mu bc} + \frac{1}{3} \tilde{A}_{abc} \right) - \psi^a \tilde{e}_a{}^\mu \tilde{A}_\mu, \tag{61}$$

We call theories I and II dual to each other if

$$\tilde{e}_a{}^\mu = f_a{}^\mu. \tag{62}$$

We then want to know whether there is a Killing-Yano tensor

$$\tilde{f}_a{}^\mu = e_a{}^\mu, \tag{63}$$

for theory II, defining a generalised supercharge for that theory[1]:

$$\tilde{Q}_f = \psi^a \tilde{f}_a{}^\mu \bar{P}_\mu + \frac{i}{2} \psi^a \psi^b \psi^c \left(\tilde{f}_a{}^\mu \tilde{\omega}_{\mu bc} + \frac{1}{3} \tilde{f}_a{}^\mu \tilde{e}_\mu{}^d \tilde{A}_{dbc} + \frac{1}{3} \tilde{c}_{abc} \right) - \psi^a \tilde{f}_a{}^\mu \tilde{A}_\mu \tag{65}$$

(with $\tilde{c}_{abc}$ defined analogous to eq.(50)). It should satisfy the relation

$$\left\{ \tilde{Q}, \tilde{Q}_f \right\}_{II} = 0. \tag{66}$$

Now eqs.(65), (66) are obtained if one takes

$$\tilde{Q} = \overline{Q_f}, \tag{67}$$

$$\tilde{Q}_f = \overline{Q}. \tag{68}$$

Indeed, eqs.(60) and (56) then immediately lead to the result

$$\left\{ \tilde{Q}, \tilde{Q}_f \right\}_{II} = \left\{ \overline{Q_f}, \overline{Q} \right\}_{II} = \overline{\{Q_f, Q\}_I} = 0. \tag{69}$$

Comparing eqs.(55) and (61) one finds the conditions

$$\tilde{e}_a{}^\mu = f_a{}^\mu, \tag{70}$$

$$3 \tilde{e}_{[a|}{}^\mu \tilde{\omega}_{\mu|bc]} + \tilde{A}_{abc} = 3 f_{[a|}{}^\mu \omega_{\mu|bc]} + f_{[a|}{}^\mu e_\mu{}^d A_{d|bc]} + c_{abc}, \tag{71}$$

[1] Notice that the transformation $\tilde{e} = f$ and $\tilde{f} = e$ is quite subtle:

$$\begin{aligned} \tilde{e}_a{}^\mu &= f_a{}^\mu \quad \Rightarrow \quad \tilde{e}_\mu{}^a = \left(f^{-1}\right)_\mu{}^a, \\ \tilde{f}_a{}^\mu &= e_a{}^\mu \quad \Rightarrow \quad \tilde{f}_\mu{}^a = \tilde{e}_\mu{}^b \tilde{f}_b{}^\nu \tilde{e}_\nu{}^a = \left(f^{-1}\right)_\mu{}^b e_b{}^\nu \left(f^{-1}\right)_\nu{}^a. \end{aligned} \tag{64}$$

$$\tilde{A}_\mu = A_\mu, \tag{72}$$

while comparison of eqs.(54) and (65) gives

$$\tilde{f}_a{}^\mu = e_a{}^\mu, \tag{73}$$

$$3\tilde{f}_{[a|}{}^\mu \tilde{\omega}_{\mu|bc]} + \tilde{f}_{[a|}{}^\mu \tilde{e}_\mu{}^d \tilde{A}_{d|bc]} + \tilde{c}_{abc} = 3e_{[a|}{}^\mu \omega_{\mu|bc]} + A_{abc}, \tag{74}$$

$$\tilde{A}_\mu = A_\mu, \tag{75}$$

Eqs.(70) and (73) are satisfied by definition and eqs.(72) and (75) are consistent. Eq.(71) defines $\tilde{A}_{abc}$ in terms of the background fields of theory I and $\tilde{\omega}_{\mu ab}$. This last quantity is fixed once $\tilde{e}_a{}^\mu$ is fixed by eq.(70). Hence we find

$$\tilde{A}_{abc} = -3f_{[a|}{}^\mu \left(\tilde{\omega}_{\mu|bc]} - \omega_{\mu|bc]}\right) + f_{[a|}{}^\mu e_\mu{}^d A_{d|bc]} + c_{abc}. \tag{76}$$

Eq.(76) then defines $\tilde{c}_{abc}$ as

$$\tilde{c}_{abc} = -3e_{[a|}{}^\mu \left(\tilde{\omega}_{\mu|bc]} - \omega_{\mu|bc]}\right) - e_{[a|}{}^\mu \left(f^{-1}\right)_\mu{}^d \tilde{A}_{d|bc]} + A_{abc}. \tag{77}$$

Hence there is always an extra supercharge $\tilde{Q}$, defined as given above, for theory II such that eq.(66) is satisfied.

One can also show this more explicitly. Working out eq.(66) one finds equations similar to (47) - (50):

$$\tilde{e}_a{}^{(\mu} \tilde{f}_b{}^{\nu)} \eta^{ab} = 0, \tag{78}$$

$$\tilde{D}^T_\mu \tilde{f}_\nu{}^a + \tilde{D}^T_\nu \tilde{f}_\mu{}^a = 0, \tag{79}$$

$$\tilde{e}_{[a}{}^\mu \tilde{f}_{b]}{}^\nu \tilde{F}_{\mu\nu} = 0, \tag{80}$$

$$\tilde{c}_{abc} = 2\tilde{e}_{[a}{}^\mu \tilde{e}_{b|}{}^\nu \tilde{D}_\nu \tilde{f}_{|c]\mu}. \tag{81}$$

Eqs.(78) and (80) are obviously equal to their dual versions in theory I and therefore automatically satisfied. Eqs.(79) and (81) are less straightforward to check but, using eqs.(47) - (50), one can indeed show that they are satisfied. Eq.(79) shows that theory II also has a Killing-Yano tensor.

Hence we have shown that a theory with a Killing-Yano tensor always has a dual theory in which the vielbein and the Killing-Yano tensor have reversed rôles. Eqs.(72) and (76) then prescribe what the electro-magnetic field and the torsion of the dual theory are. Notice that the introduction of torsion is crucial here. A theory without torsion will have a dual theory with torsion in general.

7 Kerr-Newman and Taub-NUT

Again we illustrate the general theorem derived above with the examples of the geometries of Kerr-Newman [3] and Taub-NUT [13, 14].

Kerr-Newman. First consider standard Kerr-Newman space-time, specified by the metric (15) and the Maxwell 2-form (18). In this case the torsion A_{abc} vanishes. It was found by Penrose and Floyd [11] that the Kerr-Newman metric admits a Killing-Yano tensor and hence there is an extra supersymmetry. This was described in [3] and we will give the results below.

The Kerr-Newman metric can be described by the following vierbein components:

$$\begin{aligned}
e_0{}^\mu\partial_\mu &= -\frac{1}{\rho\sqrt{\Delta}}\left[\left(r^2+a^2\right)\partial_t + a\partial_\phi\right],\\
e_1{}^\mu\partial_\mu &= \frac{\sqrt{\Delta}}{\rho}\partial_r,\\
e_2{}^\mu\partial_\mu &= \frac{1}{\rho}\partial_\theta,\\
e_3{}^\mu\partial_\mu &= \frac{1}{\rho\sin\theta}\left[a\sin^2\theta\partial_t + \partial_\phi\right].
\end{aligned} \tag{82}$$

For this geometry a Killing-Yano tensor exists, defined by

$$\begin{aligned}
f_0{}^\mu\partial_\mu &= \frac{a\sqrt{\Delta}\cos\theta}{\rho}\partial_r,\\
f_1{}^\mu\partial_\mu &= -\frac{a\cos\theta}{\rho\sqrt{\Delta}}\left[\left(r^2+a^2\right)\partial_t + a\partial_\phi\right],\\
f_2{}^\mu\partial_\mu &= \frac{r}{\rho\sin\theta}\left[a\sin^2\theta\partial_t + \partial_\phi\right],\\
f_3{}^\mu\partial_\mu &= -\frac{r}{\rho}\partial_\theta.
\end{aligned} \tag{83}$$

This Killing-Yano tensor defines an extra supercharge as given by eq.(45) with

$$\begin{aligned}
c_{012} &= -\frac{2a\sin\theta}{\rho} & c_{013} &= 0\\
c_{023} &= 0 & c_{123} &= \frac{2\sqrt{\Delta}}{\rho}
\end{aligned} \tag{84}$$

If we take f_a^μ to be the vierbein of the dual theory, then we find the dual metric given in eq.(20), in combination with a non-vanishing torsion

$$\begin{aligned} \tilde{A}_{012} &= \frac{a\sin\theta}{\rho} & \tilde{A}_{013} &= 0 \\ \tilde{A}_{023} &= 0 & \tilde{A}_{123} &= \frac{-\sqrt{\Delta}}{\rho} \end{aligned} \tag{85}$$

In accordance with eq.(75), the electro-magnetic field is the same as in the original theory. Notice that in the dual theory $dt - a\sin^2\theta d\phi$ has become space-like, while r has become time-like.

In this dual theory the original vielbein e_a^μ is a Killing-Yano tensor, defining an extra supersymmetry with non-vanishing anti-symmetric third-rank tensors

$$\begin{aligned} \tilde{c}_{012} &= 0, & \tilde{c}_{013} &= -\frac{2(3r^2 - 2a^2\cos^2\theta)\sin\theta}{3\rho r a\cos^2\theta}, \\ \tilde{c}_{023} &= -\frac{2\sqrt{\Delta}(2r^2 - 3a^2\cos^2\theta)}{3\rho r^2 a\cos\theta}, & \tilde{c}_{123} &= 0. \end{aligned} \tag{86}$$

Taub-NUT. Again we start from the standard form of the metric, eq.(21), without torsion. This metric can be described by the following vielbein

$$\begin{aligned} e_0{}^\mu\partial_\mu &= \frac{1}{\sqrt{1+\frac{2m}{r}}}\partial_r, \\ e_1{}^\mu\partial_\mu &= \frac{1}{r\sqrt{1+\frac{2m}{r}}}\partial_\theta, \\ e_2{}^\mu\partial_\mu &= \frac{1}{r\sin\theta\sqrt{1+\frac{2m}{r}}}\left[\partial_\phi - \cos\theta\,\partial_\psi\right], \\ e_3{}^\mu\partial_\mu &= \frac{\sqrt{1+\frac{2m}{r}}}{2m}\partial_\psi. \end{aligned} \tag{87}$$

The metric allows four Killing-Yano tensors [12], but three of these are trivial in the sense that they correspond to N-extended supersymmetry [14]. They are defined by

$$e_a^{(i)\mu} = M_a^{(i)b} e_b^{(0)\mu}, \qquad (i = 1, 2, 3), \tag{88}$$

where the matrices $M^{(i)}$ are of the form

$$M^{(1)} = \begin{pmatrix} 0 & \sin\phi & \cos\theta\cos\phi & -\sin\theta\cos\phi \\ -\sin\phi & 0 & -\sin\theta\cos\phi & -\cos\theta\cos\phi \\ -\cos\theta\cos\phi & \sin\theta\cos\phi & 0 & \sin\phi \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \end{pmatrix},$$

$$M^{(2)} = \begin{pmatrix} 0 & -\cos\phi & \cos\theta\sin\phi & -\sin\theta\sin\phi \\ \cos\phi & 0 & -\sin\theta\sin\phi & -\cos\theta\sin\phi \\ -\cos\theta\sin\phi & \sin\theta\sin\phi & 0 & -\cos\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi & 0 \end{pmatrix}, \tag{89}$$

$$M^{(3)} = \begin{pmatrix} 0 & 0 & -\sin\theta & -\cos\theta \\ 0 & 0 & -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta & 0 & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \end{pmatrix},$$

satisfying the quaternion algebra

$$M^{(i)}M^{(j)} = \epsilon^{ijk}M^{(k)} - \delta^{ij}\, 1. \tag{90}$$

The tensors $e_a^{(i)\mu}$ are covarianly constant and thus obey the Killing-Yano equation in a trivial way. We indicate these Killing-Yano tensors by an e in stead of an f because they define the same metric and thus the same space. We label the four different representations of this space defined by the four different vielbeine with $I = (0, i) = (0, 1, 2, 3)$. One can check explicitly that

$$D_\mu^{(I)} e_a^{(J)\nu} = 0, \qquad\qquad for\ \ I, J = 0, 1, 2, 3 \tag{91}$$

where $D_\mu^{(I)}$ is the covariant derivative in space I. Hence the $e_a^{(J)\mu}$ play the rôle of (trivial) Killing-Yano tensors in all representations I of Taub-NUT space. It is found from eq.(50) that the corresponding tensors $c_{abc}^{(I,J)}$ vanish for all J. Here we introduced a notation where the first upper index labels the space, while the second one labels the Killing-Yano tensor from which $c_{abc}^{(I,J)}$ is computed (J for $e_a^{(J)\mu}$).

Taub-NUT space admits a fourth Killing-Yano tensor ${f_a}^\mu$, which is non-trivial in this respect. In space I it is found from

$$e_a^{(I)\mu}\, f_a^{(I)\nu} = f^{\mu\nu}, \tag{92}$$

(no sum over I) where the anti-symmetric f-symbol $f^{\mu\nu}$ has the explicit form

$$f^{\mu\nu}\partial_\mu \wedge \partial_\nu = -\frac{1}{m}\partial_r \wedge \partial_\psi + \frac{2(r+m)}{rm(r+2m)\sin\theta}\partial_\theta \wedge (\partial_\phi - \cos\theta\partial_\psi)\,. \tag{93}$$

The symmetric Killing tensor (22) is the square of these Killing-Yano tensors and defines for all I the *same* dual metric (23).

The corresponding anti-symmetric third-rank tensors are found from

$$c_{abc}^{(I,\tilde{I})} = e_a^{(I)\mu} e_b^{(I)\nu} e_c^{(I)\kappa} c_{\mu\nu\kappa}. \qquad (94)$$

Again the second upper index refers to the corresponding Killing-Yano tensor: $(I, \tilde{J})$ labels the anti-symmetric three-tensor in space I computed from $f_a^{(J)\mu}$. The tensor $c_{\mu\nu\kappa}$, which is the same for all I, has only one non-vanishing component

$$c_{r\theta\phi} = \frac{2r(r+2m)\sin\theta}{m}. \qquad (95)$$

We now investigate the duality structure of this theory. The contractions

$$f^{(I,J)\mu\nu} = e_a^{(I)\mu} e_a^{(J)\nu} \qquad (96)$$

are anti-symmetric for all $I \neq J$. Together with the fact that the $e_a^{(I)\mu}$ are Killing-Yano tensors for space $I \neq J$ this implies that spaces I and J are dual to each other. The torsion in space J can then be calculated from the torsion in space I using eq.(76)

$$A_{abc}^{(J)} = -3e_{[a|}^{(J)\mu}\left(\omega_{\mu|bc]}^{(J)} - \omega_{\mu|bc]}^{(I)}\right) + e_{[a|}^{(J)\mu}\left(e^{(I)-1}\right)_\mu^{\ d} A_{d|bc]}^{(I)} + c_{abc}^{(I,J)}. \qquad (97)$$

We found already that the tensors $c_{abc}^{(I,J)}$ vanish. It is also straightforward to show, using eq.(91) for different values of I, that the spin connections $\omega_{\mu ab}^{(I)}$ are all identical. This implies that $A_{abc}^{(J)}$ is proportional to $A_{abc}^{(I)}$ for all I and J. Since the torsion vanishes in the space labeled (0), it vanishes in all spaces I, which is consistent.

Now we turn to the duality between space I and the space defined by the vielbein $f_a^{(I)\mu}$. We will label this space by $\tilde{I}$. The contraction $e_a^{(I)\mu} f_a^{(I)\nu}$ is anti-symmetric by definition (93) and furthermore $f_a^{(I)\mu}$ satisfies the Killing-Yano equation for space I. Hence spaces I and $\tilde{I}$ are dual to each other and $e_a^{(I)\mu}$ must satisfy the Killing-Yano equation in space $\tilde{I}$. The torsion in space $\tilde{I}$ is found to be defined by

$$A_{abc}^{(\tilde{I})} = f_a^{(I)\mu} f_b^{(I)\nu} f_c^{(I)\kappa} \tilde{A}_{\mu\nu\kappa}, \qquad (98)$$

with $\tilde{A}_{\mu\nu\kappa}$ having only one non-vanishing component

$$\tilde{A}_{\theta\phi\psi} = -\frac{2r^2m^2\sin\theta}{(r+m)^2}. \qquad (99)$$

Notice that $\tilde{A}_{\mu\nu\kappa}$ is the same for all spaces $\tilde{I}$. The conclusion is that there is only one space dual to Taub-NUT space. The spaces $\tilde{I}$ have the same metric and the same torsion and thus represent the same geometry.

Using the expression for the torsion we have checked explicitly that $e_a^{(I)\mu}$ satisfies the Killing-Yano equation for space $\tilde{I}$. The corresponding anti-symmetric three-tensors (space $\tilde{I}$, Killing-Yano tensor $e_a^{(I)\mu}$) are defined by

$$c_{abc}^{(\tilde{I},I)} = f_a^{(I)\mu} f_b^{(I)\nu} f_c^{(I)\kappa} \tilde{c}_{\mu\nu\kappa}, \tag{100}$$

with

$$\tilde{c}_{r\theta\phi} = \frac{2rm(r+2m)(2r^2+4rm+5m^2)\sin\theta}{3(r+m)^4}, \tag{101}$$

being the only non-vanishing component of $\tilde{c}_{\mu\nu\kappa}$, which is the same for all representations $\tilde{I}$ of dual Taub-NUT space.

The duality between Taub-NUT space and dual Taub-NUT space is not complete. The described procedure gives only one Killing-Yano tensor for dual Taub-NUT space. The $f_a^{(I)\mu}$ are not solutions of the Killing-Yano equation for space $\tilde{J}$ if $J \neq I$. And although it is easily checked that $f_a^{(J)\mu}$ satisfies the Killing-Yano equation for space I, also for $I \neq J$ (the covariant derivatives are the same in all spaces I), there is no duality between the spaces I and $\tilde{J}$ because $e_a^{(I)\mu} f_a^{(J)\nu}$ is not anti-symmetric for $I \neq J$. Hence also the $e_a^{(I)\mu}$ will not automatically be Killing-Yano tensors for space $\tilde{J}$, $J \neq I$. We have checked this explicitly and found that this is indeed not the case.

Finally we remark that in space J both $e_a^{(I)\mu}$ and $f_a^{(J)\nu}$ are Killing-Yano tensors which implies, as was shown in [3], that in that space the symmetric contraction

$$e_a^{(I)\mu} f_a^{(J)\nu} + e_a^{(I)\nu} f_a^{(J)\mu} \tag{102}$$

defines a second rank Killing tensor. For $I \neq J$ these tensors are non-trivial and are defined precisely by the conserved quantities $K^{(i)}$ given in (24) for the scalar particle. We conclude that these second rank Killing tensors are of a different type than the ones given by (19) and (22). Although they do define a dual bosonic space, they donot define a dual spinning space, since they are not the square of a Killing-Yano tensor.

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References

[1] R.H. Rietdijk and J.W. van Holten, J. of Geom. and Phys. 11 (1993), 559

[2] R.H. Rietdijk and J.W. van Holten, Class. Quantum Grav. 7 (1990), 247

[3] G.W. Gibbons, R.H. Rietdijk and J.W. van Holten, Nucl. Phys. B404 (1993), 42

[4] S.W. Hawking, Phys. Lett. A 60 (1977), 81

[5] N.S. Manton, Phys. Lett. B110 (1982), 54

[6] G.W. Gibbons and N.S. Manton, Nucl. Phys. B274(1986), 183

[7] B. Carter, Phys. Rev. D16 (1977), 3395

[8] F. De Jonghe, A.J. Macfarlane, K. Peeters and J.W. van Holten, Phys. Lett. B359 (1995), 114

[9] M. Plyushchay and P. Sodano, preprint hep-th/9511028; *id.* hep-th/9511072

[10] M. Tanimoto, Nucl. Phys. B442 (1995), 549

[11] R. Penrose, Ann. N. Y. Acad. Sci. 224 (1973), 125;
R. Floyd, *The Dynamics of Kerr Fields*, PhD. Thesis (London 1973)

[12] G.W. Gibbons and P.J. Ruback, Phys. Lett. B188 (1987), 226
Commun. Math. Phys. 115 (1988), 267

[13] M. Visinescu, Class. Quantum Grav. 11 (1994), 1867; Phys. Lett. B339 (1994), 28

[14] J.W. van Holten, Phys. Lett. B342 (1995), 47